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Harmonic families of planar curves

By

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Abstract

We establish a principle of invariance for certain weighted integrals over curves of a so-called harmonic family.

§ 1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be an open set. A function $h : \Omega \to \mathbb{R}$ with continuous second partial derivatives is called *harmonic* if $\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{\partial^2 h(x,y)}{\partial y^2} = 0$ for all $(x, y) \in \Omega$. As it is well known, such a function has the following mean value property: For every disk $\{z \in \mathbb{R}^2; |z - z_0| \leq r\} \subset \Omega$, $z_0 = (x_0, y_0) \in \mathbb{R}^2$, $r \geq 0$, it holds that

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} h(x_0 + r\cos t, y_0 + r\sin t)\,\mathrm{d}t = h(x_0, y_0)\,.$$

So, h has the same mean value over all concentric circles with center (x_0, y_0). The family of these circles is parametrized by r.

A similar phenomenon appears with respect to a family of confocal elliptic disks in Ω. Taking $(-c, 0)$ and $(c, 0)$ for their foci ($c > 0$), it holds that

$$(2) \qquad \int_0^{2\pi} h(c\cosh r\cos t, c\sinh r\sin t)\,\mathrm{d}t = 2\int_{-c}^{c} \frac{h(x,0)}{\sqrt{c^2 - x^2}}\,\mathrm{d}x$$

(see [2],[4]). So, h has the same integral over all confocal ellipses with foci at $(-c, 0)$ and $(c, 0)$. The family of these ellipses is parametrized by r.

Finally, let Ω contain a family of confocal hyperbolas (together with their interiors) with foci at $(-c, 0)$ and $(c, 0)$ ($c > 0$). If there exists $r_0 \in]0, \frac{\pi}{2}[$ such that

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$t \mapsto h(c\cos r_0 \cosh t, c\sin r_0 \sinh t)$ is integrable over $\mathbb{R}$, it then holds for $0 \le r \le r_0$ that

$$\int_{-\infty}^{\infty} h(c\cos r \cosh t, c\sin r \sinh t)\,\mathrm{d}t = 2\int_c^{\infty} \frac{h(x,0)}{\sqrt{x^2-c^2}}\,\mathrm{d}x \tag{3}$$

([5]). So, again, the left side does not change with the family parameter r, which means that h has the same integral over all these confocal hyperbolas.

Let us call a family of planar curves *harmonic* if the integral (with respect to a specified family of measures) of a harmonic function is the same over all curves of the family. It is natural to ask for a characterization of harmonic families. For this purpose we establish a general framework.

Let $I, J \subseteq \mathbb{R}$ be intervals,

$$J \times I \ni (s,t) \longmapsto (x(s,t), y(s,t)) \in \Omega$$

a mapping such that $\int_I h\left(x(s,t), y(s,t)\right) w(s,t)\,\mathrm{d}t$ is independent of $s \in J$, where w is a fixed 'weight function', and h a harmonic function for which the integral exists. The situations (1), (2), (3) belong to the special case $w = 1$, which has been treated in detail in [6]. Here we study a class of situations with nontrivial weight function w. For a family of concentric circles, the property

$$\int_0^{2\pi} h\left(\mathrm{e}^{r+\mathrm{i}t}\right) \mathrm{e}^{-kr} \cos(kt)\,\mathrm{d}t = \int_0^{2\pi} h\left(\mathrm{e}^{\mathrm{i}t}\right) \cos(kt)\,\mathrm{d}t \qquad (k \in \mathbb{N})$$

may serve as an example. It will be proven in the course of our investigation.

§ 2. Principle and equations of a harmonic family

For reasons of simplicity we develop our arguments in the case where I is a compact interval and all curves of the family are closed.

So, let $I = [a,b]$ with $a, b \in \mathbb{R}$, $a < b$. For the mapping $(s,t) \mapsto (x(s,t), y(s,t))$ we assume

$$x(s,a) = x(s,b) \quad \text{and} \quad y(s,a) = y(s,b) \tag{4}$$

for all $s \in J$.

Let $(s,t) \mapsto w(s,t)$ be the fixed (not necessarily positive) weight function, and let

$$z(s,t) := \int_a^t \frac{\partial w}{\partial s}(s,\tau)\,\mathrm{d}\tau\,. \tag{5}$$

The following theorem establishes the underlying principle of harmonic families of curves.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected open set, $J \subseteq \mathbb{R}$ a nonempty open interval, and*

$$J \times [a,b] \ni (s,t) \longmapsto (x(s,t), y(s,t)) \in \Omega \tag{6}$$

a smooth mapping satisfying

$$\frac{\partial x}{\partial s} = \frac{\partial y}{\partial t}\,, \qquad \frac{\partial x}{\partial t} = -\frac{\partial y}{\partial s}\,, \tag{7}$$

and (4). Furthermore, we assume that there exists a function $u : J \to \mathbb{R} \setminus \{0\}$ and a holomorphic function W on Ω such that

$$u(s) \cdot [z(s,t) + \mathrm{i}w(s,t)] = W\left(x(s,t) + \mathrm{i}y(s,t)\right) \tag{8}$$

for all s and t, where z and w are connected via (5). Then, for every harmonic function $h : \Omega \to \mathbb{R}$, the integral $\int_a^b h\left(x(s,t), y(s,t)\right) w(s,t)\,\mathrm{d}t$ is independent of $s \in J$.

Proof. Let $\tilde{h} : \Omega \to \mathbb{R}$ be a harmonic conjugate function to h (more precisely, let $h + \mathrm{i}\tilde{h}$ be holomorphic on Ω). It then holds:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\int_a^b h\left(x(s,t), y(s,t)\right) w(s,t)\,\mathrm{d}t \\
&\quad= \int_a^b \left[\frac{\partial h}{\partial x}\circ(x,y)\cdot\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\circ(x,y)\cdot\frac{\partial y}{\partial s}\right](s,t)w(s,t)\,\mathrm{d}t \\
&\qquad\qquad\qquad + \int_a^b h\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial s}(s,t)\,\mathrm{d}t \\
&\quad\overset{(7)}{=} \int_a^b \left[\frac{\partial \tilde{h}}{\partial y}\circ(x,y)\cdot\frac{\partial y}{\partial t} + \frac{\partial \tilde{h}}{\partial x}\circ(x,y)\cdot\frac{\partial x}{\partial t}\right](s,t)w(s,t)\,\mathrm{d}t \\
&\qquad\qquad\qquad + \int_a^b h\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial s}(s,t)\,\mathrm{d}t \\
&\quad= \int_a^b \frac{\partial}{\partial t}\tilde{h}\left(x(s,t), y(s,t)\right) w(s,t)\,\mathrm{d}t + \int_a^b h\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial s}(s,t)\,\mathrm{d}t \\
&\quad\overset{(8)}{=} -\int_a^b \tilde{h}\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial t}(s,t)\,\mathrm{d}t + \int_a^b h\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial s}(s,t)\,\mathrm{d}t \\
&\quad= \int_a^b \left[h\left(x(s,t), y(s,t)\right)\cdot\frac{\partial z}{\partial t}(s,t) - \tilde{h}\left(x(s,t), y(s,t)\right)\cdot\frac{\partial w}{\partial t}(s,t)\right]\mathrm{d}t \\
&\quad= \frac{1}{u(s)}\int_a^b \left[h\left(x(s,t), y(s,t)\right)u(s)\cdot\frac{\partial z}{\partial t}(s,t) - \tilde{h}\left(x(s,t), y(s,t)\right)u(s)\cdot\frac{\partial w}{\partial t}(s,t)\right]\mathrm{d}t \\
&\quad= \frac{1}{u(s)}\mathrm{Re}\int_a^b \left[\left(h + \mathrm{i}\tilde{h}\right)\circ(x,y)\right](s,t)\cdot\frac{\partial(uz + \mathrm{i}uw)}{\partial t}(s,t)\,\mathrm{d}t\,,
\end{aligned}$$

where 'Re' denotes the real part (we keep the letter u for the function $(s,t) \mapsto u(s)$),

$$= -\frac{1}{u(s)} \mathrm{Re} \int_a^b \frac{\partial \left(h + \mathrm{i}\tilde{h}\right) \circ (x,y)}{\partial t}(s,t) \cdot (uz + \mathrm{i}uw)(s,t)\, \mathrm{d}t\,.$$

If γ_s denotes the curve $t \mapsto (x(s,t), y(s,t))$, this is further equal to

$$-\frac{1}{u(s)} \mathrm{Re} \int_{\gamma_s} \left(h + \mathrm{i}\tilde{h}\right)' \cdot W\,, \tag{9}$$

where $\left(h + \mathrm{i}\tilde{h}\right)'$ denotes the complex derivative of the holomorphic function $h + \mathrm{i}\tilde{h}$. From our conditions it follows that $\left(h + \mathrm{i}\tilde{h}\right)' \cdot W$ is a holomorphic function on Ω and γ_s a closed curve in Ω. Therefore, (9) vanishes, and the theorem is proven. $\square$

The main condition in the theorem is the conformality (7) of the mapping $(s,t) \mapsto (x(s,t), y(s,t))$. As a consequence of it, there exists a harmonic function $v = v(s,t)$ on $J \times [a,b]$ such that

$$x = \frac{\partial v}{\partial t}\,, \quad y = \frac{\partial v}{\partial s}\,. \tag{10}$$

Conversely, if (10) holds with a harmonic function v, then $(s,t) \mapsto (x(s,t), y(s,t))$ is clearly conformal. So, a harmonic v can give rise to a harmonic family of curves, provided that (8) is satisfied with a holomorphic function W.

Usually the starting point is a given curve $[a,b] \ni t \mapsto (x_0(t), y_0(t)) \in \Omega$ together with a given weight function $[a,b] \ni t \mapsto w_0(t)$. The goal is then to determine a harmonic family that contains this curve, i. e. to find a mapping $(s,t) \mapsto (x(s,t), y(s,t))$ and a function $(s,t) \mapsto w(s,t)$ satisfying the conditions of the theorem such that

$$x(0,t) = x_0(t)\,, \quad y(0,t) = y_0(t)\,, \quad \text{and} \quad w(0,t) = w_0(t)$$

for all $t \in [a,b]$ (without loss of generality we assume that $0 \in J$).

For the harmonic function v, this implies that

$$\frac{\partial v}{\partial t}(0,t) = x_0(t)\,, \quad \frac{\partial v}{\partial s}(0,t) = y_0(t)\,. \tag{11}$$

Thus, v satisfies a Cauchy problem with initial data (11) and is therefore determined up to an additive constant. From (11) and the harmonicity of v it inductively follows that

$$\begin{aligned} \frac{\partial^{2k} v}{\partial s^{2k}}(0,t) &= (-1)^k x_0^{(2k-1)}(t) \quad \text{for } k \in \mathbb{N}\,, \\ \frac{\partial^{2k+1} v}{\partial s^{2k+1}}(0,t) &= (-1)^k y_0^{(2k)}(t) \quad \text{for } k \in \mathbb{N} \cup \{0\}\,. \end{aligned}$$

This leads to the series expansion

$$v(s,t) = v(0,t) + \sum_{k=1}^{\infty} \frac{(-1)^k x_0^{(2k-1)}(t)}{(2k)!} s^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k y_0^{(2k)}(t)}{(2k+1)!} s^{2k+1}, \tag{12}$$

on the basis of which, together with $\frac{\partial v(0,t)}{\partial t} = x_0(t)$, (10) constitutes the equations of the corresponding harmonic family. The s-interval of convergence of (12) depends on the initial curve (x_0, y_0). Furthermore, it follows from (10) and (12) that if (x_0, y_0) is a **closed** curve and x_0, y_0 can be extended to **smooth periodic** functions on $\mathbb{R}$, then $[a,b] \ni t \mapsto (x(s,t), y(s,t))$ remains closed for every $s \in J$. At this point it is obvious that the initial closed curve has to be smooth, otherwise there is a problem with condition (4). Of course, it still remains to be checked that condition (8) can be satisfied. Here, there seem to be many possibilities, since only the limiting values of w are prescribed: $\lim_{s\to 0} w(s,t) = w_0(t)$.

A function v as above (which is determined up to a constant) will be called a *d-potential* (from the word *deformation*) of the curve (x_0, y_0) and the corresponding harmonic family. This should not be confused with the Schwarz potential of (x_0, y_0) (if this exists; see [3]).

§ 3. Specification of the weight function

In this section we shall examine the consequences on the form of the weight function w that follow from condition (8), W being a holomorphic function there.

Since $s + \mathrm{i}t \mapsto x(s,t) + \mathrm{i}y(s,t)$ is holomorphic because of (7), the left side of (8) should be a holomorphic function of $s + \mathrm{i}t$. Denoting by a prime the derivative with respect to s, by a dot the one with respect to t, we obtain:

$$\begin{cases} u'z + uz' = u\dot{w} \\ u'w + uw' = -u\dot{z} \end{cases} \Longleftrightarrow \begin{cases} \frac{u'}{u} = \frac{\dot{w} - z'}{z} \\ \frac{u'}{u} = -\frac{w' + \dot{z}}{w} \end{cases}. \tag{13}$$

From (5), $\dot{z} = w'$, so we have $\frac{u'(s)}{u(s)} = -2 \cdot \frac{w'(s,t)}{w(s,t)}$. Taking $\alpha(s) := -\frac{u'(s)}{2u(s)}$ and

$$A(s) := \int_0^s \alpha(s')\,\mathrm{d}s' = -\frac{1}{2} \ln \frac{u(s)}{u(0)}$$

it follows that w is of the form $w(s,t) = d(t) \cdot \mathrm{e}^{A(s)}$.

Let $D(t) := \int_a^t d(\tau)\,\mathrm{d}\tau$. From (5) it follows that

$$z(s,t) = \int_a^t d(\tau) \cdot \alpha(s) \cdot \mathrm{e}^{A(s)} \mathrm{d}\tau = \alpha(s) \mathrm{e}^{A(s)} D(t)\,. \tag{14}$$

Inserting all these expressions into the first equation of the right system of (13) we obtain:

$$
\begin{aligned}
-\,2\alpha(s) &= \frac{\dot d(t)\mathrm{e}^{A(s)} - \alpha'(s)\mathrm{e}^{A(s)}D(t) - \alpha(s)^2\mathrm{e}^{A(s)}D(t)}{\alpha(s)\mathrm{e}^{A(s)}D(t)} \\
\iff -\alpha(s) &= \frac{\dot d(t)}{\alpha(s)D(t)} - \frac{\alpha'(s)}{\alpha(s)} \\
\iff \frac{\ddot D(t)}{D(t)} &= \alpha'(s) - \alpha(s)^2 =: -\mu \in \mathbb{R}\,. \qquad (15)
\end{aligned}
$$

Here we have to distinguish three cases.

§ 3.1. $\mu > 0$

From (15) and the definition of $D(t)$ it follows that

$$D(t) = c_0 \sin\left(\sqrt{\mu}(t-a)\right)$$

with $c_0 \in \mathbb{R}\setminus\{0\}$, and $\alpha'(s) = \alpha(s)^2 - \mu$.

If $\alpha(s) \equiv \pm\sqrt{\mu}$, this would result in $u(s) \equiv c_1 \cdot \mathrm{e}^{\mp 2\sqrt{\mu}s}$ with $c_1 \in \mathbb{R}\setminus\{0\}$ and

$$d(t) = \dot D(t) = c_0\sqrt{\mu}\cos\left(\sqrt{\mu}(t-a)\right)\,,$$

so $w(s,t) = c_0\sqrt{\mu}\mathrm{e}^{\pm\sqrt{\mu}s}\cos\big(\sqrt{\mu}(t-a)\big)$, $z(s,t) = \pm c_0\sqrt{\mu}\mathrm{e}^{\pm\sqrt{\mu}s}\sin\big(\sqrt{\mu}(t-a)\big)$, and

$$u(s)\cdot[z(s,t) + \mathrm{i}w(s,t)] = \mathrm{i}c_1c_0\sqrt{\mu}\mathrm{e}^{\pm\mathrm{i}a\sqrt{\mu}}\cdot\mathrm{e}^{\mp\sqrt{\mu}(s+\mathrm{i}t)}\,. \qquad (16)$$

If $\alpha(s) \not\equiv \pm\sqrt{\mu}$, we obtain:

$$
\begin{aligned}
\frac{\alpha'(s)}{\alpha(s)^2-\mu} = 1 \iff & \frac{\alpha'(s)}{\alpha(s)-\sqrt{\mu}} - \frac{\alpha'(s)}{\alpha(s)+\sqrt{\mu}} = 2\sqrt{\mu} \\
\iff & \ln\left|\frac{\alpha(s)-\sqrt{\mu}}{\alpha(s)+\sqrt{\mu}}\right| = 2\sqrt{\mu}s + c_2\,, \quad c_2 \in \mathbb{R} \\
\iff & \frac{\alpha(s)-\sqrt{\mu}}{\alpha(s)+\sqrt{\mu}} = c_3\mathrm{e}^{2\sqrt{\mu}s}, \quad c_3 \in \mathbb{R}\setminus\{0,1\} \\
\iff & \alpha(s) = \frac{\sqrt{\mu}\mathrm{e}^{-\sqrt{\mu}s} + c_3\mathrm{e}^{\sqrt{\mu}s}\sqrt{\mu}}{\mathrm{e}^{-\sqrt{\mu}s} - c_3\mathrm{e}^{\sqrt{\mu}s}} \\
\iff & A(s) = \ln\frac{1-c_3}{\mathrm{e}^{-\sqrt{\mu}s} - c_3\mathrm{e}^{\sqrt{\mu}s}} \\
\iff & u(s) = c_4\cdot\left(c_3\mathrm{e}^{\sqrt{\mu}s} - \mathrm{e}^{-\sqrt{\mu}s}\right)^2, \quad c_4 \in \mathbb{R}\setminus\{0\}\,.
\end{aligned}
$$

So,

$$w(s,t) = c_0\sqrt{\mu}\cos\left(\sqrt{\mu}(t-a)\right)\cdot\frac{1-c_3}{\mathrm{e}^{-\sqrt{\mu}s} - c_3\mathrm{e}^{\sqrt{\mu}s}}\,. \qquad (17)$$

Moreover, after the computations and up to real multiplicative constants, the left side of (8) becomes

$$u(s)\cdot[z(s,t)+\mathrm{i}w(s,t)]=\mathrm{i}\mathrm{e}^{\mathrm{i}a\sqrt{\mu}}\cdot\mathrm{e}^{-\sqrt{\mu}(s+\mathrm{i}t)}-\mathrm{i}c_3\mathrm{e}^{-\mathrm{i}a\sqrt{\mu}}\mathrm{e}^{\sqrt{\mu}(s+\mathrm{i}t)}\,. \tag{18}$$

The expressions (16), (18), being holomorphic in $s+\mathrm{i}t$, the existence of a holomorphic W such that (8) holds depends on the mapping (6), so the further examination differs from case to case. However, there still is a further general specification possible. On the basis of (4) and (8) it should hold $w(s,a)=w(s,b)$ and $z(s,a)=z(s,b)$ for all $s\in J$. This means that $\sqrt{\mu}(b-a)=2k\pi$ with $k\in\mathbb{N}$, so $\mu=\frac{4\pi^2k^2}{(b-a)^2}$.

§ 3.2. $\mu<0$

In this case we have $D(t)=c_0\sinh\left(\sqrt{-\mu}(t-a)\right)$, which has to be rejected, since it is not compatible with $w(s,a)=w(s,b)$ due to (4).

§ 3.3. $\mu=0$

In this case $D(t)$ is an affine function, say $D(t)=c_0(t-a)$, $c_0\in\mathbb{R}\setminus\{0\}$. Since $z(s,a)=z(s,b)$ because of (4), from (14) it follows that $\alpha(s)\equiv 0$. So, u is constant, and so is the weight function w. This situation has already been studied in [6].

§ 4. Applications

(I) Our first example concerns concentric circles. Here,

$$x(s,t)=\mathrm{e}^s\cos t\,,\quad y(s,t)=\mathrm{e}^s\sin t$$

(by $x+\mathrm{i}y=\mathrm{e}^{s+\mathrm{i}t}$ this is clearly a holomorphic mapping; here, $a=0$ and $b=2\pi$). According to (16) and (18) on the basis of (8) we have the possibilities $\mathrm{e}^{-k(s+\mathrm{i}t)}=W(\mathrm{e}^{s+\mathrm{i}t})$, $\mathrm{e}^{k(s+\mathrm{i}t)}=W(\mathrm{e}^{s+\mathrm{i}t})$, or $\mathrm{e}^{-k(s+\mathrm{i}t)}-c_3\mathrm{e}^{k(s+\mathrm{i}t)}=W(\mathrm{e}^{s+\mathrm{i}t})$. Clearly, only $\mathrm{e}^{k(s+\mathrm{i}t)}=W(\mathrm{e}^{s+\mathrm{i}t})$ is possible with a holomorphic W on a simply connected set.

It follows that $w(s,t)$ is proportional to $\mathrm{e}^{-ks}\cos(kt)$, so the integral

$$\int_0^{2\pi}h\left(\mathrm{e}^s\cos t,\mathrm{e}^s\sin t\right)\mathrm{e}^{-ks}\cos(kt)\,\mathrm{d}t$$

is independent of s. This is exactly what had been mentioned in the introduction.

(II) For a family of confocal ellipses with foci at $(-c,0)$ and $(c,0)$ $(c>0)$ we take

$$x(s,t)=c\cosh s\cos t\,,\quad y(s,t)=c\sinh s\sin t\,.$$

Since $x+\mathrm{i}y=c\cosh(s+\mathrm{i}t)$, this is clearly a holomorphic mapping. Again, $a=0$ and $b=2\pi$.

Apart from (2), there also exists the following mean value property:

(19)
$$\frac{1}{\pi(a^2+b^2)}\int_0^{2\pi} h(a\cos t, b\sin t)(a^2\sin^2 t + b^2\cos^2 t)\,\mathrm{d}t = \frac{2}{\pi c^2}\int_{-c}^{c} h(x,0)\sqrt{c^2-x^2}\,\mathrm{d}x$$

having put $a := c\cosh s$ and $b := c\sinh s$ (left and right side of (19) equal 1 for $h = 1$). Let us give a short proof of it.

In [1] it was proven that

$$\int_{E_s} h\,\mathrm{d}\lambda = \sinh(2s)\cdot\int_{-1}^{1} h(x,0)\sqrt{1-x^2}\,\mathrm{d}x\,,$$

where E_s denotes the elliptic disk $\frac{x^2}{\cosh^2 s} + \frac{y^2}{\sinh^2 s} \leq 1$ and λ the Lebesgue measure on it. By a change of coordinates we have

$$\int_{E_s} h\,\mathrm{d}\lambda = \int_0^s\int_0^{2\pi} h(\cosh r\cos t, \sinh r\sin t)(\cosh^2 r\sin^2 t + \sinh^2 r\cos^2 t)\,\mathrm{d}t\,\mathrm{d}r\,.$$

If we differentiate these two equations with respect to s and then connect their right sides, we arrive at the asserted mean value property after a similarity transformation.

Our purpose here is to show that (2) and (19) are special cases of a family of mean value properties that our theorem provides.

According to (16) and (18) there should exist a holomorphic function W on a simply connected domain Ω such that $W(\cosh(s+\mathrm{i}t)) = \mathrm{e}^{k(s+\mathrm{i}t)}$ or $W(\cosh(s+\mathrm{i}t)) = \mathrm{e}^{-k(s+\mathrm{i}t)}$ or $W(\cosh(s+\mathrm{i}t)) = \mathrm{e}^{-k(s+\mathrm{i}t)} - c_3\mathrm{e}^{k(s+\mathrm{i}t)}$. Since cosh is an even function, the only possibility is the third one with $c_3 = -1$, that is, up to a multiplicative constant,

$$W(\cosh(s+\mathrm{i}t)) = \cosh(k(s+\mathrm{i}t))\,.$$

That such a holomorphic W indeed exists follows by induction on $k \in \mathbb{N}\cup\{0\}$. By (17), up to multiplicative constants, $w(s,t) = \frac{\cos(kt)}{\cosh(ks)}$, so the integral

$$\int_0^{2\pi} h\left(c\cosh s\cos t, c\sinh s\sin t\right)\cdot\frac{\cos(kt)}{\cosh(ks)}\,\mathrm{d}t$$

is independent of s. Taking $k = 2$ we obtain

(20)
$$\int_0^{2\pi} h(c\cosh s\cos t, c\sinh s\sin t)\cdot\frac{\cos(2t)}{\cosh(2s)}\,\mathrm{d}t = \int_0^{2\pi} h(c\cos t, 0)\cos(2t)\,\mathrm{d}t$$

for s in an open interval J containing 0. On the other hand, $k = 0$ leads to

(21)
$$\int_0^{2\pi} h(c\cosh s\cos t, c\sinh s\sin t)\,\mathrm{d}t = \int_0^{2\pi} h(c\cos t, 0)\,\mathrm{d}t\,,$$

which is equivalent to (2). From (20) and (21) we obtain

$$\int_0^{2\pi} h(c\cosh s\cos t, c\sinh s\sin t)\left(\frac{1}{2}-\frac{\cos(2t)}{2\cosh(2s)}\right)\mathrm{d}t = \int_0^{2\pi} h(c\cos t, 0)\sin^2 t\,\mathrm{d}t$$

and finally

$$\frac{1}{\pi}\int_0^{2\pi} h(c\cosh s\cos t, c\sinh s\sin t)\left(\frac{1}{2}-\frac{\cos(2t)}{2\cosh(2s)}\right)\mathrm{d}t = \frac{2}{\pi c^2}\int_{-c}^{c} h(x,0)\sqrt{c^2-x^2}\,\mathrm{d}x.$$

Now it is a simple computation to see that this is nothing else but (19).

(III) Integral formulae may be possible even if W is not everywhere holomorphic. For $s > \ln 4$ let

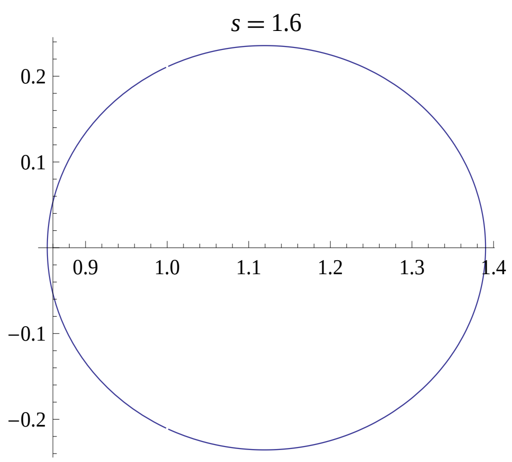
$$x(s,t) = \frac{\alpha_s(t)\sqrt{\mathrm{e}^{2s}\cos(2t)+4\mathrm{e}^s\cos t+\mathrm{e}^s\sqrt{\mathrm{e}^{2s}+16+8\mathrm{e}^s\cos t}}}{2\sqrt{2}} - \frac{\mathrm{e}^s\cos t}{2},$$

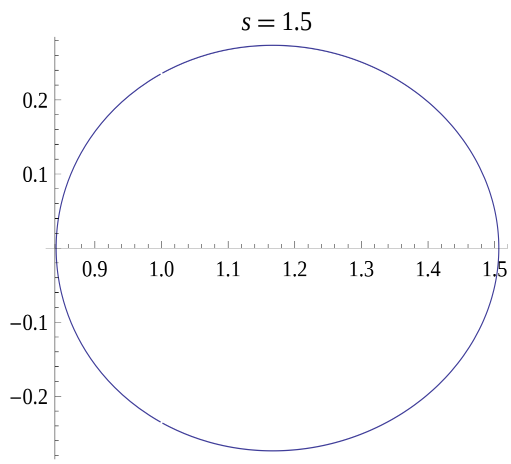
$$y(s,t) = \frac{\mathrm{e}^{2s}\sin(2t)+4\mathrm{e}^s\sin t}{2\sqrt{2}\alpha_s(t)\sqrt{\mathrm{e}^{2s}\cos(2t)+4\mathrm{e}^s\cos t+\mathrm{e}^s\sqrt{\mathrm{e}^{2s}+16+8\mathrm{e}^s\cos t}}} - \frac{\mathrm{e}^s\sin t}{2},$$

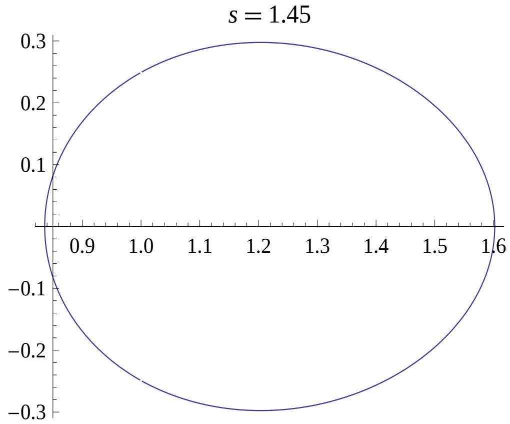
where

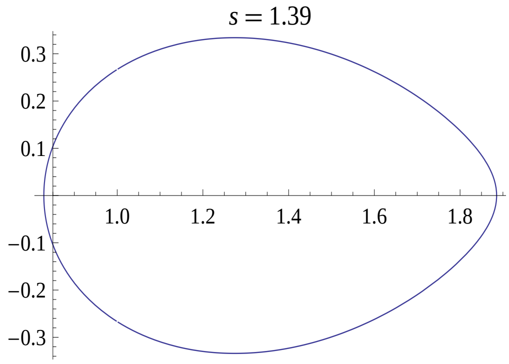
$$\alpha_s(t) = \begin{cases} 1 \text{ for } 0 \le t < \arccos\left(-\frac{2}{\mathrm{e}^s}\right) \\ -1 \text{ for } \arccos\left(-\frac{2}{\mathrm{e}^s}\right) \le t < 2\pi - \arccos\left(-\frac{2}{\mathrm{e}^s}\right) \\ 1 \text{ for } 2\pi - \arccos\left(-\frac{2}{\mathrm{e}^s}\right) \le t \le 2\pi \end{cases}.$$

The following figures show these curves for some values of s.









Taking $W(z) = \frac{\mathrm{i}z^2}{1-z}$ we have $W\left(x(s,t) + \mathrm{i}y(s,t)\right) = \mathrm{i}\mathrm{e}^{s+\mathrm{i}t}$. (The equation $\frac{z^2}{1-z} = a$ has a solution which depends holomorphically on a outside the segment (interval) $[-4, 0]$. Since $a = \mathrm{e}^{s+\mathrm{i}t}$, this explains the restriction $s > \ln 4$. A lengthy computation leads to the above expressions $x(s,t)$ and $y(s,t)$ for the solution $z = x(s,t) + \mathrm{i}y(s,t)$. The adjustment of the sign with the help of $\alpha_s(t)$ is necessary for smoothness.)

So, we can take $w(s,t) = \mathrm{e}^{-s}\cos t$, $u(s) = \mathrm{e}^{2s}$ (see the beginning of subsection 3.1). However, W has a simple pole at $z = 1$, and this point lies in the interior of each curve γ_s in the proof of the theorem. Following the reasoning there, by the residue theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^{2\pi} h\left(x(s,t), y(s,t)\right)\mathrm{e}^{-s}\cos t\,\mathrm{d}t = -2\pi\mathrm{e}^{-2s}\mathrm{Re}\left[\partial_1 h(1,0) + \mathrm{i}\partial_1\tilde{h}(1,0)\right]$$

$$= -2\pi\mathrm{e}^{-2s}\partial_1 h(1,0)\,,$$

where ∂_1 denotes the differentiation with respect to the first variable. By integration we obtain

$$\int_0^{2\pi} h\left(x(s,t), y(s,t)\right)\mathrm{e}^{-s}\cos t\,\mathrm{d}t = \pi\mathrm{e}^{-2s}\partial_1 h(1,0) + c$$

with a constant $c \in \mathbb{R}$ that depends on the function h.

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